Suggested solution of HW5

1. (a) Suppose $\limsup \frac{x_{n+1}}{x_n} = r < 1$. Then there is N such that

$$\sup_{k\geq N} \frac{x_{k+1}}{x_k} < \frac{1+r}{2} = R < 1.$$

Hence, for all $n \in \mathbb{N}$,

$$x_{n+N} \leq Rx_{n+N-1} \leq R^n x_N.$$

Therefore, for all $m < n$

$$\begin{aligned} 0 < \sum_{k=N+m}^{n+N} x_k &= \sum_{k=m}^{n} x_{k+N} \\ &\leq x_N \sum_{k=m}^{n} R^k \\ &\leq \frac{x_N R^m}{1-R}. \end{aligned}$$

Hence, $\{s_n = \sum_{k=1}^n x_k\}$ is a cauchy sequence implying the convergence.

(b) There is N such that

$$\inf_{k\geq N} \frac{x_{k+1}}{x_k} \geq r = \frac{1+R}{2} > 1.$$

Thus, for all $n \in \mathbb{N}$,

$$x_{n+N} \geq rx_{n+N-1} \geq r^n x_N.$$

As $r^n \to \infty$, by divergence test, the series is divergent.

2. Suppose $\limsup_n \frac{x_{n+1}}{x_n} = L \neq \infty$. Let $\epsilon > 0$, there is N such that for all $n \geq N$

$$x_{n+1} \leq (L+\epsilon)x_n \leq (L+\epsilon)^{n-N+1} x_N.$$

Hence,

$$x_{n+1}^{\frac{1}{n+1}} \leq (L+\epsilon)^{\frac{n+1-N}{n+1}} x_N^{\frac{1}{n+1}}.$$

Since the inequality holds for all $n \geq N$, we may let $n \to \infty$ which immediately obtain

$$\limsup_n x_n^{\frac{1}{n}} \leq L + \epsilon.$$

ϵ is arbitrary. Hence.

$$\limsup_n x_n^{\frac{1}{n}} \leq \limsup_n \frac{x_{n+1}}{x_n}.$$

If $L = \infty$, there is nothing to prove. The lower bound is similar.